THE MATHEMATICAL ANTINOMIES AND THEIR SOLUTION.

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If we suppose two parallel straight lines, unlimited in extent, and intersected by perpendiculars drawn at equal distances from each other, since it is evident that each division upon the one line is equal to each division upon the other, and that any number of divisions upon the one will equal in extent a corresponding number upon the other, the question naturally arises whether the equation will not hold good when all the divisions are considered. Whether the lines may not be regarded as equal in extent, and whether the sum of the divisions upon both lines will not be equal to twice the sum of the divisions upon either line alone? That is, are we not forced to conclude that one infinite may be equal to, less, or greater than another?

In the correct answer to this question lies the solution of the mathematical antinomies, which have their origin in a false conception of the infinite, and are in no sense contradictions into which the reason, legitimately used, must fall. The fallacy contained in the above reasoning is palpable. It is true that we must consider each division on the one line equal to each division on the other, and, taking any number of divisions on the one and adding them to an equal number on the other, we obtain a sum equal to twice the number of given divisions on either. But when we say "*all* the divisions on the one are equal to *all* the divisions on the other," we speak of the lines as quantitative wholes, and introduce an error with the word *all.* To conceive of a thing as a whole, we must assign to it limits, and in saying "the whole" of any object we refer to those limits beyond which there is none of that object. In regarding any object as a quantitative whole, we necessarily think it as finite. When we compare one line with another and say that its extent is greater or less than that of the other, we mean that, when the one is applied to the other, its limits extend beyond or fall within those of the other. In other words, we give the difference between the distances included between their respective limits. Measuring is merely giving the distance between

limits. In the case of the two infinite lines we have no point to measure from, and no point to measure to, and no measurement—therefore no comparison is possible. It is a palpable contradiction to compare (*i. e.*, give relations of measurement between the respective limits of) two infinites (*i. e.*, things which cannot be measured as having no limits).

The terms longer, shorter, equal, can therefore have no meaning as applied to infinite lines, and are legitimately used only in speaking of the finite.

As a line can only be increased by adding to it at its extremities, it is manifestly absurd to speak of the sum of the two lines mentioned above as greater than either line alone; but there are cases in which the error of a wrong conclusion is not so immediately palpable—as, for example, the case of a line limited at but one point. May we not here add to the line at its extremity, and thus increase its total length? At first glance it would seem so, but when we recollect that the line is limited only at one point, and is not, therefore, as a line, defined (for two points are necessary to define a line), the impossibility of regarding it as a quantitative whole is evident, and the impossibility of increasing or diminishing its length *as a whole* necessarily follows. The word "all" cannot be applied to the line either in its original state or after it has been added to. The question, therefore, whether a line without any limits is not greater than one which is limited at one point, is rightly answered by saying that the very nature of the conceptions precludes the possibility of the words "greater" or "less" being applied to either; that neither of the lines can be regarded quantitatively, and that, consequently, the question is a meaningless one.

The reasoning here applied to lines will also apply to surfaces and solids. It is unnecessary to multiply instances, as the principle is in all cases the same. In general, wherever the limit is removed in any one direction, whether in the case of lines, surfaces, or solids, the object can no longer be regarded as finite, and, consequently, not as a quantitative whole.

If we use the word infinite in its strict etymological sense, as referring to a total absence of limits, that which has even one limit cannot, of course, be called infinite. We find such a use of the word in the writings of Sir William Hamilton, who asserts

that past time, since it is bounded by the present, cannot be infinite—"a bounded infinite is a contradiction."[1] But arguments drawn from the etymological signification of a word are valueless, unless that signification expresses the true and whole content of the word. That such is not the case here is evident. A line limited at but one point is certainly not finite, for it cannot be regarded as a whole, cannot be increased, diminished, or compared with other lines; in short, it is not subject to the conditions of the finite. If, then, for etymological reasons, we exclude it from the class of infinites, we have the infinite, the finite, and a *tertium quid*, which is between the two. There is, however, no difficulty in classing such a line with the infinite, for they are subject to the same conditions, and equally distinct from the finite.

It remains to consider a class of cases of an apparently different nature from those we have examined. It is argued that an infinite series of dollars will exceed in value an infinite series of cents—that, where the unit differs, the difference will extend to the series in its totality. The error of such an assumption may be easily shown by showing what the assertion necessarily involves.

Suppose that, instead of counting one cent in the one series to each dollar in the other, we vary our mode of procedure by counting one hundred cents in the one to each dollar in the other. It is true that the one series will be exhausted one hundred times as rapidly as the other; but, since they are both infinite (will never end), we may continue thus forever (to infinity). We may then regard the two series as of equal value. And, by successively changing the unit, we may make the one series greater than, equal to, or less than the other, the value depending merely on the mode of reckoning. If we have a right to make an estimate of the comparative values of the series in the first instance, we have the same right in the second, as the error in the two is identical, and consists in regarding an infinite series as a whole, capable as a whole of increase or diminution. An infinite cannot be made one member of an equation, for, having abstracted the quantitative, we have abstracted the condition under which alone an equation is valid, and the form becomes meaningless.

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[1] "Metaph.," Boston, 1859, pp. 527 *et seq.*

The difficulties which will arise from overlooking this important fact are well instanced in that agnostic theory which Sir William Hamilton developed under the name of the Philosophy of the Conditioned, the fundamental principle of which is that "all which is conceivable in thought lies between two extremes, which, as contradictory of each other, cannot both be true, but of which, as mutual contradictories, one must."[1]

Let us examine his application of this law to our conception of space:

"We are altogether unable to conceive space as bounded—as finite; that is, as a whole, beyond which there is no further space. Every one is conscious that this is impossible. . . . The one contradictory is thus found inconceivable; we cannot conceive space as absolutely limited.

"On the other hand, we are equally powerless to realize in thought the possibility of the opposite contradictory; we cannot conceive space as infinite, as without limits. You may launch out in thought beyond the solar walk, you may transcend in fancy even the universe of matter, and rise from sphere to sphere in the region of empty space, until imagination sinks exhausted; with all this what have you done? You have never gone beyond the finite, you have attained at best only to the indefinite, and the indefinite, however expanded, is still always the finite. . . ."

That the former of these contradictories is inconceivable we must admit; but the argument used to prove the latter inconceivable is plainly faulty. We may, indeed, "rise from sphere to sphere in the region of empty space" without transcending the finite; we cannot arrive at the unlimited while we carry our limits with us. Each successive stage simply places the limits farther apart, and in no respect tends to do away with them altogether. This attempt to arrive at the infinite forcibly reminds one of the tragical history of the amusing person in Chamisso's poem, who supposed that, by turning quickly around, he could cause his cue to hang in front.

> "Er dreht sich links, er dreht sich rechts,
> Es thut nichts Gut's, es thut nichts Schlecht's—
> Der Zopf, der hängt ihm hinten."

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[1] "Metaph.," Boston, 1859, pp. 527 *et seq.*

And how analogous would be the condition of one who would still seek to reach the infinite by endlessly continuing this hopeless journey to that of the hero as portrayed in the last verse!

> "Und seht, er dreht sich immer noch,
> Und denkt: es hilft am Ende doch—
> Der Zopf, der hängt ihm hinten."

It is not by adding space to space that we arrive at the idea of infinite space. Imagination may well "sink exhausted" in the attempt to find the end of the limitless. This is an attempt to realize infinite space as a quantitative whole, and, so considered, it is manifestly inconceivable, as containing a contradiction. The antinomies arising from the consideration of the minimum of space, and those which have to do with our idea of time, are equally capable of solution by the substitution of the true (qualitative) idea of the infinite for the quantitative idea; the error is in all cases identical, and the contradiction a gratuitous one.

It is interesting to notice that that acutest of thinkers, Immanuel Kant, although he has based the proof of the thesis of his first antinomy on a false conception of the infinite, and although, after correctly criticising the false conception, he himself lapses into it, yet perceived, and in so many words gave expression to the fact, that the conception of the infinite is not a quantitative one.

The thesis of the first antinomy maintains that the world had a beginning in time, and is limited with regard to space—both of which are denied in the antithesis. The proofs offered in support of the antithesis may be passed over as extraneous to the subject; those in support of the thesis I will quote, not for the purpose of again pointing out their fallacious character, for they are identical with the arguments used by Sir William Hamilton, but in order that I may give the observations appended to them, which are significant in their contextual connection. The proof proceeds by assuming the truth of the antithesis, and then proving it to be impossible:

"Granted, that the world has no beginning in time; up to every given moment of time an eternity must have elapsed, and therewith passed away an infinite series of successive conditions or states of things in the world. Now, the infinity of a series consists in the fact that it never can be completed by means of a suc-

cessive synthesis. It follows that an infinite series already elapsed is impossible, and that, consequently, a beginning of the world is a necessary condition of its existence. And this was the first thing to be proved.

"As regards the second, let us take the opposite for granted. In this case the world must be an infinite given total of coexistent things. Now, we cannot cogitate the dimensions of a quantity, which is not given within certain limits of an intuition, in any other way than by means of a synthesis of its parts, and the total of such a quantity only by means of a completed synthesis, or the repeated addition of unity to itself. Accordingly, to cogitate the world, which fills all spaces, as a whole, the successive synthesis of the parts of an infinite world must be looked upon as completed—that is to say, an infinite time must be regarded as having elapsed in the enumeration of all coexisting things, which is impossible. For this reason an infinite aggregate of actual things cannot be considered as a given whole, consequently not as a contemporaneously given whole. The world is, consequently, as regards extension in space, *not infinite*, but enclosed in limits. And this was the second thing to be proved."[1]

It will be noticed that the word *completed* (vollendet) is used in the first part of the proof in a manner to which we may object as misleading. When we speak of a series as "completed by means of a successive synthesis," we are apt to regard it as a whole, having a beginning as well as an end. The inconsequent nature of the reasoning in the latter part of the proof it is scarcely necessary to point out. The observations on the thesis are the following:

"In bringing forward these conflicting arguments, I have not been on the search for sophisms, for the purpose of availing myself of special pleading, which takes advantage of the carelessness of the opposite party, appeals to a misunderstood statute, and erects its unrighteous claims upon an unfair interpretation. Both proofs originate fairly from the nature of the case, and the advantage presented by the mistakes of the dogmatists of both parties has been completely set aside.

"The thesis might also have been unfairly demonstrated by the introduction of an erroneous conception of the infinity of a given quantity. A quantity is infinite if a greater than itself cannot

[1] "Critique." Trans. by Meiklejohn. London, 1876, pp. 266

possibly exist. The quantity is measured by the number of given units—which are taken as a standard—contained in it. Now, no number can be the greatest, because one or more units can always be added. It follows that an infinite given quantity, consequently an infinite world (both as regards time and extension), is impossible. It is therefore limited in both respects. In this manner I might have conducted my proof; but the conception given in it does not agree with the true conception of an infinite whole. In this there is no representation of its quantity; it is not said how large it is; consequently, its conception is not the conception of a *maximum*. We cogitate in it merely its relation to an arbitrarily assumed unit, in relation to which it is greater than any number. Now, just as the unit which is taken is greater or smaller, the infinite will be greater or smaller; but the infinity, which consists merely in the relation to this given unit, must remain always the same, although the absolute quantity of the whole is not thereby cognized.

"The true (transcendental) conception of infinity is: that the successive synthesis of unity in the measurement of a given quantum can never be completed. Hence it follows, without possibility of mistake, that an eternity of actual successive states up to a given (the present) moment cannot have elapsed, and that the world must, therefore, have a beginning.

"In regard to the second part of the thesis, the difficulty as to an infinite and yet elapsed series disappears; for the manifold of a world infinite in extension is contemporaneously given. But, in order to cogitate the total of this manifold, as we cannot have the aid of limits constituting by themselves this total in intuition, we are obliged to give some account of our conception, which in this case cannot proceed from the whole to the determined quantity of the parts, but must demonstrate the possibility of a whole by means of a successive synthesis of the parts. But as this synthesis must constitute a series that cannot be completed, it is impossible for us to cogitate prior to it, and, consequently, not by means of it, a totality. For the conception of totality itself is, in the present case, the representation of a completed synthesis of the parts; and this completion, and, consequently, its conception, is impossible." [1]

[1] "Critique," pp. 268 ff.

We here find a conception of the infinite brought forward as false; a declaration of wherein it differs from the true conception; and a statement of what, according to Kant, the true conception really is. "A quantity is infinite if a greater than itself cannot possibly exist." We can readily see that such a conception gives us, not an infinite, but a finite. Not only is the word *greater* inapplicable to infinites, but the very expression "a *quantity* is infinite" is absurd, as involving a contradiction. Kant was too clear a thinker not to see that that which admits of an addition of units, and consequently of increase as a whole, cannot be infinite. He declares that this does not agree with the true conception of the infinite, in which "there is no representation of its quantity, it is not said how large it is; consequently its conception is not the conception of a *maximum*." This is a clear recognition of the fact that the conception cannot be quantitative.

But it is evident that Kant did not see the full force and the logical consequences of this statement. In the sentence immediately preceding he uses the phrase "an infinite whole," and in the sentences immediately following he brings forward a conception faulty in precisely the same respect as the one criticised. "We cogitate in it merely its relation to an arbitrarily assumed unit, in relation to which it is greater than any number. Now, just as the unit which is taken is greater or smaller, the infinite will be greater or smaller; but the infinity, which consists merely in the relation to this given unit, must remain always the same, although the absolute quantity of the whole is not thereby cognized." That is, if we designate the infinite by a, the unit by b, and the infinity (the relation of a to b) by x, we find that a varies as b, but that x remains always the same (and this can only mean *numerically* the same).

The infinity is, in this case, simply an indefinite number, and the quantity of the whole can certainly be cognized. The error is identical with that in the case just cited, and both parts of the proof given in support of the thesis of the first antinomy will fall to the ground when this error is rectified.

It remains to consider a case which apparently militates against the theory that an infinite series can never be regarded as a whole. In the case of a point moving uniformly along a line, over the whole of which it will pass in a given time, we have a descending

series which we may assume to be represented by $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}. \; . \; . \; . \; 0$. The point will have moved over one half of the line in half a minute, over one fourth more in a quarter of a minute, etc., until, when the minute is completed, the point will have arrived at zero. We find here, under a slightly different guise, the old problem of Achilles and the Tortoise. Must we not regard the whole series as contained between the two limits 1 and 0, and capable of completion by a successive synthesis?

A moment's consideration will reveal the fallacy of such a mode of reasoning. The series is not completed at all, but is truly infinite. It is limited at one point by the the highest member, $\frac{1}{2}$; but is not limited at another by the zero, since this can only be assumed as a limit to the series by breaking the law of the series, which is that each term shall be half as great as the term preceding. We can never, by halving something, arrive at nothing; a division of substance will never give us that which is not substance. The 0, since it does not make one in the series, cannot limit the series. The error lies in regarding the series as capable of completion by passing through all degrees of the composite to the simple, and from that to 0 as a final term. Whether we hold to the Kantian conception of space, or to the Berkeleyan, which would deny to any given portion of space an infinite divisibility, our conclusions will be the same as to the impossibility of the completion of an infinite series. According to the former, space and time are composites. A space is made up of spaces, and never by subdividing it could we arrive at that which is not space. The point in question passes over the whole line, not by completing the descending series until it arrives at the simple, but by the successive addition of spaces, which are composites. A line is not made up of points, for a point is possible only as the limit of a line. If one point has no extension, a thousand will have no more. We cannot, by multiplying points, create in them a property which no single point can possess in the slightest degree. "As space is not a composite of substances (and not even of real accidents), if I abstract all composition therein, nothing—not even a point—remains; for a point is possible only as the limit of a space—consequently, of a composite. Space and time, therefore, do not consist of simple parts."[1]

[1] "Critique," p. 274.

We cannot, therefore, consider any member of the series under consideration as the smallest possible, but must regard the series as truly endless. We have, then, an infinite series, limited at but one point, which cannot be regarded as a sum total, a quantitative whole, equal as a whole to the given line; and the apparent exception we find to be not incompatible with the general position we have assumed.

According to the Berkeleyan theory, which would hold that the subdivision of any given portion of space will result in the simple, we are compelled to assume that the point in question passes over the line by the successive addition of simple parts; but we may still hold the mathematical series to be infinite. The negation of an infinite divisibility to space does not imply the negation of the infinity of a mathematical series, but simply implies that mathematical reasonings can be applied to the determination of space only within certain limits—those of a possible perception. We find, then, that, on either theory, this antinomy, like all the others, depends upon a misconception, and is capable of an easy solution.

FACTS OF CONSCIOUSNESS.

TRANSLATED FROM THE GERMAN OF J. G. FICHTE BY A. E. KROEGER.

Part Third.—*Concerning the Higher Faculty.*

Chapter III.

GENERAL REVIEW OF ALL THE PRECEDING.

Life, as One, is simply because it is; and in this its Being it is altogether not an object of contemplation, but an object of thinking; and, moreover, of pure thinking, or intellectualizing.

It cannot be contemplated, for contemplation is a being of immediate freedom. But life in its pure being is not free at all to tear itself loose from that being; it is absolutely tied down to that its formal being. It is, therefore, absolutely impossible that life should have an immediate contemplation of its being.

Nevertheless, it is thinkable. It has freedom to manifest itself